

NONLINEAR NEUMANN PROBLEMS DRIVEN BY A NONHOMOGENEOUS DIFFERENTIAL OPERATOR

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ABSTRACT. We study a nonlinear parametric Neumann problem driven by a nonhomogeneous quasilinear elliptic differential operator $\operatorname{div}(a(x, \nabla u))$, a special case of which is the p -Laplacian. The reaction term is a nonlinearity function f which exhibits $(p - 1)$ -subcritical growth. By using variational methods, we prove a multiplicity result on the existence of weak solutions for such problems. An explicit example of an application is also presented.

1. INTRODUCTION

In this paper we study the existence of multiple solutions for the following Neumann problem,

$$(N_{\lambda, \mu}) \quad \begin{cases} -\operatorname{div}(a(x, \nabla u)) + |u|^{p-2}u = (\lambda k(x) + \mu)f(u) & \text{in } \Omega \\ \frac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here and in the sequel, Ω is a bounded, connected domain in $(\mathbb{R}^N, |\cdot|)$ with smooth boundary $\partial\Omega$, $p > 1$, $a : \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a suitable Carathéodory map which is strictly monotone in the $\xi \in \mathbb{R}^N$ variable and $\partial u / \partial n_a := a(x, \nabla u) \cdot n$, where n is the outward unit normal vector on $\partial\Omega$. Further, λ and μ are positive real parameters, $k \in L^\infty(\Omega)_+$ and finally, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which is $(p - 1)$ -sublinear at infinity. We cite a recent monograph by Kristály, Rădulescu and Varga [13] as a general reference on variational methods.

Recently, problems involving p -Laplacian-like operators have been studied by several authors under different boundary conditions and by using different technical approaches.

For instance, Dirichlet problems involving a general operator in divergence form were studied by De Nápoli and Mariani in [6] by imposing symmetry condition on the map $\xi \mapsto a(a, \xi)$. In the cited paper the existence of one weak solution was proved by exploiting the standard mountain pass geometry and requiring, among other assumptions, that the nonlinearity f has a $(p - 1)$ -superlinear behaviour at infinity. The non-uniform case was successively considered by Duc and Vu in [7] who extended the result of [6] under the key hypothesis that the map a fulfills a suitable growth condition.

In [12], by using variational methods, Kristály, Lisei and Varga studied the analogue of the above case for a uniform Dirichlet problem with parameter, obtaining

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the existence of three weak solutions requiring that the nonlinearity f has a $(p-1)$ -sublinear growth at infinity.

Successively, Yang, Geng and Yan [26] proved the existence of three weak solutions for singular p -Laplacian type equations. Finally, Papageorgiou, Rocha and Staicu in [21] considered a nonsmooth p -Laplacian problem in divergence form, obtaining the existence of at least two nontrivial weak solutions. See also the contributions obtained by Servadei in [23] for related multiplicity results.

The study of the corresponding Neumann problem is in some sense lagging behind. Superlinear Neumann problems were studied by Aizicovici, Papageorgiou and Staicu [1] and Gasiński-Papageorgiou [10]. In [1] the differential operator is the p -Laplacian and the superlinear reaction term satisfies the celebrated (AR)-condition. In [10] the differential operator is nonhomogeneous incorporating the p -Laplacian, but for the superlinear case the authors prove only an existence theorem and do not have multiplicity results. Related to this paper are also the nice works [11, 20] and references therein.

Our goal in this paper is to prove a multiplicity result for Neumann problem $(N_{\lambda,\mu})$ by using a critical point result due to Ricceri (see Theorem 2.1). More precisely, for a suitable $\mu = \mu_0$ and λ sufficiently small, the existence of multiple solutions for problem (N_{λ,μ_0}) will be obtained requiring that the nonlinearity f has a $(p-1)$ -linear growth in addition to a suitable oscillating behaviour of the associated potential (see condition $(h_m^{\mu_0})$). We also emphasize that our hypotheses on a , following the approach given in [10], are considerably weaker than the corresponding ones in [6, 12], where $a(x, \xi) =: \nabla_\xi A(x, \xi)$, with $A \in C(\bar{\Omega} \times \mathbb{R}^N)$ and for every $x \in \bar{\Omega}$, $A(x, \cdot) \in C^1(\mathbb{R}^N)$. Moreover, they assume that for every $x \in \bar{\Omega}$, the function $\xi \mapsto A(x, \xi)$ is a strongly convex function.

This requirement, in the special case of the p -Laplacian operator $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ implies that $p \geq 2$. In contrast, in our approach we only have that for every $x \in \bar{\Omega}$, the map $\xi \mapsto A(x, \xi)$ is strictly convex. So, for p -Laplacian equations we allow any $p > 1$.

The plan of the paper is as follows. Section 2 is devoted to our abstract framework, while Section 3 is dedicated to the main results. A concrete example of an application is then presented (see Example 3.8).

2. ABSTRACT FRAMEWORK

Let $W^{1,p}(\Omega)$ ($p > 1$) be the usual Sobolev space, equipped with the norm

$$\|u\| := \left(\int_{\Omega} (|\nabla u(x)|^p + |u(x)|^p) dx \right)^{1/p}.$$

Further, let $W^{-1,p'}(\Omega)$, with $1/p + 1/p' = 1$, be its topological dual and denote the duality brackets for the pair $(W^{-1,p'}(\Omega), W^{1,p}(\Omega))$ by $\langle \cdot, \cdot \rangle$. Indicate by p^* the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.

Recall that if $p < N$ then $p^* = Np/(N-p)$ and for every $q \in [1, p^*]$ there exists a positive constant c_q such that

$$(1) \quad \|u\|_{L^q(\Omega)} \leq c_q \|u\|,$$

for every $u \in W^{1,p}(\Omega)$. Moreover, when $p \geq N$, this inequality holds for any $q \in [1, +\infty[$, since $p^* = +\infty$.

Our main tool will be the following abstract critical point theorem due to Ricceri [22].

Theorem 2.1. *Let H be a separable and reflexive real Banach space and let $\mathcal{N}, \mathcal{G} : H \rightarrow \mathbb{R}$ be sequentially weakly lower semicontinuous and continuously Gâteaux differentiable functionals, with \mathcal{N} coercive.*

Assume that the functional $J_\lambda := \mathcal{N} + \lambda \mathcal{G}$ satisfies the Palais-Smale condition for every $\lambda > 0$ small enough and that the set of all global minima of \mathcal{N} has at least m connected components in the weak topology, with $m \geq 2$.

Then for every $\eta > \inf_{u \in H} \mathcal{N}(u)$, there exists $\bar{\lambda} > 0$ such that for every $\lambda \in (0, \bar{\lambda})$, the functional J_λ has at least $m + 1$ critical points, m of which are lying in the set $\mathcal{N}^{-1}((-\infty, \eta))$.

For the sake of completeness, we also recall that a C^1 -functional $J : X \rightarrow \mathbb{R}$, where X is a real Banach space with topological dual X^* , satisfies the Palais-Smale condition at level $\mu \in \mathbb{R}$, (briefly $(PS)_\mu$) when

$(PS)_\mu$ Every sequence $\{u_n\}$ in X such that

$$J(u_n) \rightarrow \mu, \quad \text{and} \quad \|J'(u_n)\|_{X^*} \rightarrow 0,$$

possesses a convergent subsequence.

Finally, we say that J satisfies the Palais-Smale condition (in short (PS)) if $(PS)_\mu$ holds for every $\mu \in \mathbb{R}$.

3. MAIN RESULT

In the sequel, let $\Omega \subset \mathbb{R}^N$ be a bounded and connected Euclidean domain. Assume that there exists a function $A : \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}$, with gradient $a(x, \xi) := \nabla_\xi A(x, \xi) : \bar{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$, such that the following conditions hold:

- (α_1) For all $\xi \in \mathbb{R}^N$, the function $x \mapsto A(x, \xi)$ is measurable;
- (α_2) For almost all $x \in \bar{\Omega}$, the function $\xi \mapsto A(x, \xi)$ is C^1 , strictly convex, and $A(x, 0) = 0$;
- (α_3) For almost all $x \in \bar{\Omega}$ and all $\xi \in \mathbb{R}^N$, we assume

$$|a(x, \xi)| \leq a_0(x) + c_0 |\xi|^{p-1},$$

with $a_0 \in L^\infty(\Omega)_+$, $c_0 > 0$ and $p > 1$;

- (α_4) For almost all $x \in \bar{\Omega}$ and all $\xi \in \mathbb{R}^N$, we suppose

$$a(x, \xi) \cdot \xi \leq pA(x, \xi);$$

- (α_5) There exists $\kappa > 0$ such that for almost all $x \in \bar{\Omega}$ and every $\xi \in \mathbb{R}^N$, we have $\kappa |\xi|^p \leq pA(x, \xi)$.

Example 3.1. We present some examples of functions $A(x, \xi)$ which correspond to the map $a(x, \xi)$ and satisfy the above hypotheses.

- $A(x, \xi) := \frac{|\xi|^p}{p}$ with $p > 1$. Then

$$a(x, \xi) := \nabla_\xi A(x, \xi) = |\xi|^{p-2} \xi.$$

In this setting, the resulting differential operator is the usual p -Laplacian

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u);$$

- $A(x, \xi) := \frac{a_1(x)}{p}|\xi|^p + \frac{a_2(x)}{p}|\xi|^r$, with $a_1, a_2 \in L^\infty(\Omega)_+$, $a_1(x) \geq c_0 > 0$ for almost every $x \in \bar{\Omega}$ and $1 < r < p$;
- $A(x, \xi) := \frac{a_1(x)}{p}|\xi|^p + \frac{1}{r} \log(1 + |\xi|^r)$, with $a_1 \in L^\infty(\Omega)_+$, $a_1(x) \geq c_0 > 0$ for almost every $x \in \bar{\Omega}$ and $1 < r \leq p$;
- $A(x, \xi) := \frac{1}{p}((1 + |\xi|^2)^{p/2})$, with $p > 1$. Thus

$$a(x, \xi) = (1 + |\xi|^2)^{(p-2)/2} \xi.$$

The resulting differential operator is the generalized mean curvature operator

$$\operatorname{div}((1 + |\nabla u|^2)^{(p-2)/2} \nabla u);$$

- $A(x, \xi) := \frac{M(x)\xi \cdot \xi}{2}$, with $M \in L^\infty(\bar{\Omega}; \mathbb{R}^{N \times N})$ and $M(x) \geq c_0 I_N$ for almost every $x \in \bar{\Omega}$, with $c_0 > 0$ and I_N being the identity N -matrix.

Remark 3.2. The operator $a(x, \xi) := \nabla_\xi A(x, \xi)$ satisfies the (S_+) property; see [10, Proposition 3.1]. This means that for every sequence $\{u_n\} \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and

$$\limsup_{n \rightarrow \infty} \int_{\Omega} a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x) dx \leq 0,$$

then $u_n \rightarrow u$ (strongly) in $W^{1,p}(\Omega)$.

From now on, let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that

$$(h_\infty) \quad \lim_{|t| \rightarrow \infty} \frac{f(t)}{|t|^{p-1}} = 0.$$

A typical case when (h_∞) holds is

$$(h_{q\rho}) \quad \text{There exist } q \in (0, p-1) \text{ and } \rho > 0 \text{ such that } |f(t)| \leq \rho |t|^q \text{ for every } t \in \mathbb{R}.$$

In order to obtain our multiplicity result, in addition to condition (h_∞) , we also require that:

$$(h_m^{\mu_0}) \quad \text{There exists } \mu_0 \in (0, \infty) \text{ such that the set of global minima of the function}$$

$$s \mapsto \tilde{F}_{\mu_0}(s) := \Lambda s^p - \mu_0 F(s),$$

has at least $m \geq 2$ connected components.

Note that $(h_m^{\mu_0})$ implies that the function $s \mapsto \tilde{F}_{\mu_0}(s)$ has at least $m - 1$ local maxima.

We are interested in the existence of multiple weak solutions for the following Neumann problem

$$(N_{\lambda, \mu_0}) \quad \begin{cases} -\operatorname{div}(a(x, \nabla u)) + |u|^{p-2}u = (\lambda k(x) + \mu_0)f(u) & \text{in } \Omega \\ \frac{\partial u}{\partial n_a} = 0 & \text{on } \partial\Omega. \end{cases}$$

For the sake of completeness we recall that, fixing $\lambda > 0$, a *weak solution* of problem (N_{λ, μ_0}) is a function $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} a(x, \nabla u(x)) \cdot \nabla v(x) dx &= - \int_{\Omega} |u(x)|^{p-2} u(x) v(x) dx \\ &+ \lambda \int_{\Omega} k(x) f(u(x)) v(x) dx \\ &+ \mu_0 \int_{\Omega} f(u(x)) v(x) dx, \end{aligned}$$

for every $v \in W^{1,p}(\Omega)$.

Set $\Phi : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ given by

$$\Phi(u) := \int_{\Omega} A(x, \nabla u(x)) dx + \frac{1}{p} \int_{\Omega} |u(x)|^p dx,$$

and

$$\mathcal{N}_{\mu_0}(u) := \Phi(u) - \mu_0 \int_{\Omega} F(u(x)) dx,$$

as well as

$$\mathcal{G}(u) := - \int_{\Omega} k(x) F(u(x)) dx,$$

for every $u \in W^{1,p}(\Omega)$. Here, as usual, we put

$$F(s) := \int_0^s f(t) dt,$$

for every $s \in \mathbb{R}$.

With the above notations and assumptions, it is easy to prove that \mathcal{N}_{μ_0} and \mathcal{G} are C^1 -functionals with Gâteaux derivatives given by

$$\begin{aligned} \langle \mathcal{N}'_{\mu_0}(u), v \rangle &= \int_{\Omega} a(x, \nabla u(x)) \cdot \nabla v(x) dx + \int_{\Omega} |u(x)|^{p-2} u(x) v(x) dx \\ &- \mu_0 \int_{\Omega} f(u(x)) v(x) dx, \end{aligned}$$

and

$$\langle \mathcal{G}'(u), v \rangle = - \int_{\Omega} k(x) f(u(x)) v(x) dx,$$

for every $v \in W^{1,p}(\Omega)$.

Thus, the critical points of $J_{\lambda} := \mathcal{N}_{\mu_0} + \lambda \mathcal{G}$ are exactly the weak solutions of problem (P_{λ}) .

Finally, denote

$$\Lambda := \frac{\min \{\kappa, 1\}}{p}.$$

Standard arguments ensure the validity of the following preliminary regularity result on the functionals \mathcal{N}_{μ_0} and \mathcal{G} .

Lemma 3.3. *Let us assume that condition (h_{∞}) holds. Then the above functionals \mathcal{N}_{μ_0} and \mathcal{G} are sequentially weakly lower semicontinuous.*

Proof. Due to condition (α_2) the functional Φ is convex. Since Φ is strongly continuous it is also weakly lower semicontinuous. On the other hand, since condition (h_∞) holds, there exists a positive constant c such that $|f(t)| \leq c(1 + |t|^{p-1})$, for every $t \in \mathbb{R}$. Finally, due to the fact that the embedding $X \hookrightarrow L^p(\Omega)$ is compact, we obtain that the functionals

$$u \mapsto - \int_{\Omega} F(u(x)) dx, \quad \text{and} \quad u \mapsto - \int_{\Omega} k(x) F(u(x)) dx,$$

are sequentially weakly lower semicontinuous by arguing in standard way. \square

Further, the C^1 -functional J_λ satisfies the (PS)-condition as proved in the next result.

Lemma 3.4. *Assume that condition (h_∞) holds. Then the functional J_λ is coercive and satisfies the (PS)-condition for every real parameter λ .*

Proof. Let us fix $\lambda \in \mathbb{R}$ and consider

$$0 < \alpha < \frac{1}{\mu_0 + |\lambda| \|k\|_\infty}.$$

By condition (h_∞) , there exists δ_λ such that

$$|f(t)| \leq \frac{\alpha p \Lambda}{c_p^p} |t|^{p-1},$$

for every $|t| \geq \delta_\lambda$. By integration we have

$$|F(s)| \leq \frac{\alpha \Lambda}{c_p^p} |s|^p + \max_{|t| \leq \delta_\lambda} |f(t)| |s|,$$

for every $s \in \mathbb{R}$.

Thus, by using the above inequality and bearing in mind relation (1), one has

$$\begin{aligned} J_\lambda(u) &\geq \Phi(u) - \mu_0 \left| \int_{\Omega} F(u(x)) dx \right| - |\lambda| |\mathcal{G}(u)| \\ &\geq \Lambda(1 - \alpha(\mu_0 + |\lambda| \|k\|_\infty)) \|u\|^p \\ &\quad - c_1(\mu_0 + |\lambda| \|k\|_\infty) \max_{|t| \leq \delta_\lambda} |f(t)| \|u\|, \end{aligned}$$

where $p' := (p-1)/p$ is, as usual, the conjugate exponent of p . Then the functional J_λ is bounded from below and, since $p > 1$, $J_\lambda(u) \rightarrow +\infty$ whenever $\|u\| \rightarrow +\infty$. Hence J_λ is coercive.

Now, fix $\mu \in \mathbb{R}$ and let us prove that J_λ satisfy the condition $(PS)_\mu$. For this goal, let $\{u_n\} \subset W^{1,p}(\Omega)$ be a Palais-Smale sequence, i.e.

$$J_\lambda(u_n) \rightarrow \mu, \quad \text{and} \quad \|J'_\lambda(u_n)\|_{W^{-1,p'}} \rightarrow 0.$$

Taking into account the coercivity of J_λ , the sequence $\{u_n\}$ is necessarily bounded in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is reflexive, we may extract a subsequence that for simplicity we call again $\{u_n\}$, such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$.

We will prove that u_n strongly converges to $u \in W^{1,p}(\Omega)$. Exploiting the derivative $J'_\lambda(u_n)(u_n - u)$, we obtain

$$\begin{aligned} \int_{\Omega} a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x) dx &= \langle J'_\lambda(u_n), u_n - u \rangle \\ &- \int_{\Omega} |u_n(x)|^{p-2} u_n(x) (u_n - u)(x) dx \\ &- \mu_0 \int_{\Omega} f(u_n(x)) (u_n - u)(x) dx \\ &- \lambda \int_{\Omega} k(x) f(u_n(x)) (u_n - u)(x) dx. \end{aligned}$$

Since $\|J'_\lambda(u_n)\|_{W^{-1,p}} \rightarrow 0$ and the sequence $\{u_n - u\}$ is bounded in $W^{1,p}(\Omega)$, taking into account that $|\langle J'_\lambda(u_n), u_n - u \rangle| \leq \|J'_\lambda(u_n)\|_{W^{-1,p'}} \|u_n - u\|$, one has

$$\langle J'_\lambda(u_n), u_n - u \rangle \rightarrow 0.$$

Further, by the asymptotic condition (h_∞) , there exists a real positive constant c such that $|f(t)| \leq c(1 + |t|^{p-1})$, for every $t \in \mathbb{R}$. Then

$$\begin{aligned} &\int_{\Omega} |f(u_n(x))| |u_n(x) - u(x)| dx \\ &\leq c \left(\int_{\Omega} |u_n(x) - u(x)| dx + \int_{\Omega} |u_n(x)|^{p-1} |u_n(x) - u(x)| dx \right) \\ &\leq c((\text{meas}(\Omega))^{1/p'} + \|u_n\|_{L^p}^{p-1}) \|u_n - u\|_{L^p}. \end{aligned}$$

Now, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, hence $u_n \rightarrow u$ strongly in $L^p(\Omega)$. So we obtain

$$\int_{\Omega} |f(u_n(x))| |u_n(x) - u(x)| dx \rightarrow 0.$$

Analogously, one has

$$\int_{\Omega} k(x) |f(u_n(x))| |u_n(x) - u(x)| dx \rightarrow 0.$$

Moreover, considering the inequality

$$\begin{aligned} \int_{\Omega} |u_n(x)|^{p-2} u_n(x) (u_n(x) - u(x)) dx &= \int_{\Omega} |u_n(x)|^{p-1} |u_n(x) - u(x)| dx \\ &\leq \|u_n\|_{L^p}^{p-1} \|u_n - u\|_{L^p}, \end{aligned}$$

and $u_n \rightarrow u$ strongly in $L^p(\Omega)$, we have

$$\int_{\Omega} |u_n(x)|^{p-2} u_n(x) (u_n(x) - u(x)) dx \rightarrow 0.$$

We can conclude that

$$\limsup_{n \rightarrow \infty} \langle a(x, u_n), u_n - u \rangle \leq 0,$$

where $\langle a(x, u_n), u_n - u \rangle$ denotes

$$\int_{\Omega} a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x) dx.$$

But as observed in Remark 3.2, the operator Φ' has the (S_+) property. So, in conclusion, $u_n \rightarrow u$ strongly in $W^{1,p}(\Omega)$.

Hence, J_λ is bounded from below and fulfills (PS), for every positive parameter λ . \square

Remark 3.5. We observe that by the above lemma, the functional

$$J_0 = \mathcal{N}_{\mu_0}(u) := \Phi(u) - \mu_0 \int_{\Omega} F(u(x))dx, \quad (u \in W^{1,p}(\Omega))$$

is coercive.

Proposition 3.6. *The set of all global minima of the functional \mathcal{N}_{μ_0} has at least m connected components in the weak topology on $W^{1,p}(\Omega)$.*

Proof. First, for every $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned} \mathcal{N}_{\mu_0}(u) &= \Phi(u) - \mu_0 \int_{\Omega} F(u(x))dx \\ &\geq \Lambda \int_{\Omega} |\nabla u(x)|^p dx + \int_{\Omega} \tilde{F}_{\mu_0}(u(x))dx \\ &\geq \left(\inf_{s \in \mathbb{R}} \tilde{F}_{\mu_0}(s) \right) \text{meas}(\Omega). \end{aligned}$$

Moreover, if we consider $u(x) = u_{\tilde{s}}(x) = \tilde{s}$ for almost every $x \in \Omega$, where $\tilde{s} \in \mathbb{R}$ is a minimum point of the function $s \mapsto \tilde{F}_{\mu_0}(s)$, then we have the equality from the previous estimate (note that $\Phi(0) = 0$ by using the last part of condition (α_2)). Thus,

$$\inf_{u \in W^{1,p}(\Omega)} \mathcal{N}_{\mu_0}(u) = \left(\inf_{s \in \mathbb{R}} \tilde{F}_{\mu_0}(s) \right) \text{meas}(\Omega).$$

Further, if $u \in W^{1,p}(\Omega)$ is not a constant function, we have

$$\begin{aligned} \mathcal{N}_{\mu_0}(u) &\geq \Lambda \int_{\Omega} |\nabla u(x)|^p dx + \int_{\Omega} \tilde{F}_{\mu_0}(u(x))dx \\ &> \left(\inf_{s \in \mathbb{R}} \tilde{F}_{\mu_0}(s) \right) \text{meas}(\Omega). \end{aligned}$$

Consequently, between the sets

$$\text{Min}(\mathcal{N}_{\mu_0}) = \left\{ u \in W^{1,p}(\Omega) : \mathcal{N}_{\mu_0}(u) = \inf_{u \in W^{1,p}(\Omega)} \mathcal{N}_{\mu_0}(u) \right\},$$

and

$$\text{Min}(\tilde{F}_{\mu_0}) = \left\{ s \in \mathbb{R} : \tilde{F}_{\mu_0}(s) = \inf_{s \in \mathbb{R}} \tilde{F}_{\mu_0}(s) \right\},$$

there is a one-to-one correspondence.

Indeed, let θ be the function which associates to every number $s \in \mathbb{R}$ the equivalence class of those functions which are almost everywhere equal to s in Ω .

Then $\theta : \text{Min}(\tilde{F}_{\mu_0}) \rightarrow \text{Min}(\mathcal{N}_{\mu_0})$ is actually a homeomorphism between $\text{Min}(\tilde{F}_{\mu_0})$ and $\text{Min}(\mathcal{N}_{\mu_0})$, where the set $\text{Min}(\mathcal{N}_{\mu_0})$ is considered with the relativization of the weak topology on $W^{1,p}(\Omega)$.

On account of the hypothesis $(h_m^{\mu_0})$, the set $\text{Min}(\tilde{F}_{\mu_0})$ contains at least $m \geq 2$ connected components. Therefore, the same is true for the set $\text{Min}(\mathcal{N}_{\mu_0})$, which completes the proof. \square

Our main result is as follows.

Theorem 3.7. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that conditions (h_∞) and $(h_m^{\mu_0})$ hold. Then*

- a) *For every $\eta > 0$, there exists a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ problem (N_λ) has at least $m+1$ weak solutions $u_\lambda^{1,\eta}, \dots, u_\lambda^{m+1,\eta} \in W^{1,p}(\Omega)$; and*
- b) *If $(h_{q\rho})$ holds then for each $\lambda \in (0, \tilde{\lambda}_\eta)$ there is a set $I_\lambda \subset \{1, \dots, m+1\}$ with $\text{card}(I_\lambda) = m$ such that*

$$\|u_\lambda^{i,\eta}\| < t_{\eta q\rho}, \quad (i \in I_\lambda)$$

where $t_{\eta q\rho} > 0$ is the greatest solution of the equation

$$\Lambda t^p - \rho\mu_0 \frac{\text{meas}(\Omega)^{((p-1)-q)/p}}{q+1} t^{q+1} - \eta = 0, \quad (t > 0).$$

Proof. Let us choose $H = W^{1,p}(\Omega)$, and

$$\mathcal{N} := \mathcal{N}_{\mu_0} = \Phi(u) - \mu_0 \int_{\Omega} F(u(x)) dx,$$

as well as

$$\mathcal{G}(u) := - \int_{\Omega} k(x) F(u(x)) dx,$$

for every $u \in W^{1,p}(\Omega)$, in Theorem 2.1.

Due to Proposition 3.6, Lemmas 3.3 and 3.4 all the hypotheses of Theorem 2.1 are satisfied.

Note that $\mathcal{N}(0) = 0$, so $\inf_{u \in H} \mathcal{N}(u) \leq 0$. Therefore, for every

$$\eta > 0 \geq \inf_{u \in H} \mathcal{N}(u),$$

there is a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ the function $\mathcal{N}_{\mu_0} + \lambda \mathcal{G}$ has at least $m+1$ critical points; let us denote them by $u_\lambda^{1,\eta}, \dots, u_\lambda^{m+1,\eta} \in H$. Clearly, they are solutions of problem (N_λ) , which proves the first claim.

We know in addition that m elements from $u_\lambda^{1,\eta}, \dots, u_\lambda^{m+1,\eta}$ belong to the set $\mathcal{N}_{\mu_0}^{-1}((-\infty, \eta))$. Let \tilde{u} be such an element, i.e.,

$$\mathcal{N}_{\mu_0}(\tilde{u}) = \Phi(\tilde{u}) - \mu_0 \int_{\Omega} F(\tilde{u}(x)) dx < \eta.$$

Hence, one has

$$(2) \quad \Lambda \|\tilde{u}\|^p - \mu_0 \int_{\Omega} F(\tilde{u}(x)) dx < \eta.$$

Assume that $(h_{q\rho})$ holds. Then $|F(t)| \leq \frac{\rho}{q+1} |t|^{q+1}$ for every $t \in \mathbb{R}$.

By using the Hölder inequality, one has

$$(3) \quad \int_{\Omega} |\tilde{u}(x)|^{q+1} dx \leq \text{meas}(\Omega)^{((p-1)-q)/p} \|\tilde{u}\|^{q+1}.$$

On account of (2) and (3) it follows that

$$(4) \quad \Lambda \|\tilde{u}\|^p - \rho\mu_0 \frac{\text{meas}(\Omega)^{((p-1)-q)/p}}{q+1} \|\tilde{u}\|^{q+1} < \eta.$$

Now, observe that, since $\eta > 0$ and $q \in (0, p-1)$, it is easy to see that the following algebraic equation

$$(5) \quad \Lambda t^p - \rho \mu_0 \frac{\text{meas}(\Omega)^{((p-1)-q)/p}}{q+1} t^{q+1} - \eta = 0,$$

always has a positive solution.

Finally, bearing in mind (4), the number $\|\tilde{u}\|$ is less than the greatest solution $t_{\eta q p} > 0$ of the equation (5). The proof is complete. \square

In conclusion we present a direct and easy application of Theorem 3.7 for an elliptic Neumann problem involving the Laplace operator.

Example 3.8. Let $k \in L^\infty(\Omega)_+$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be the continuous function defined by $f(t) := \min\{t_+ - \sin(\pi t_+), 2(m-1)\}$, where $m \geq 2$ is fixed and $t_+ = \max\{t, 0\}$. Consider the following Neumann problem

$$(\tilde{N}_{\lambda,1}) \quad \begin{cases} -\Delta u + u = (\lambda k(x) + 1)f(u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Owing to Theorem 3.7, for every $\eta > 0$, there exists a number $\tilde{\lambda}_\eta > 0$ such that for every $\lambda \in (0, \tilde{\lambda}_\eta)$ problem $(\tilde{N}_{\lambda,1})$ has at least $m+1$ weak solutions $u_\lambda^{1,\eta}, \dots, u_\lambda^{m+1,\eta} \in W^{1,2}(\Omega)$. Indeed, clearly, (h_∞) holds, while for $\mu_0 = 1$, the assumption (h_m^1) is also fulfilled. Indeed, the function $t \mapsto \tilde{F}_1(t)$ has precisely m global minima; they are $0, 2, \dots, 2(m-1)$. Moreover, $\min_{t \in \mathbb{R}} \tilde{F}_1(t) = 0$.

Remark 3.9. We emphasize that there are several multiplicity results for nonlinear Neumann problems driven by the p -Laplacian differential operator. We mention, among others, the works [2, 3, 5, 9, 19]. With exception of [5] and [19], in all the cited papers, it is assumed that $p > N$ and the authors exploit the fact that, in this context, the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$.

Remark 3.10. For completeness we also cite a recent interesting paper of Colasuonno, Pucci, and Varga [4] which contains some multiplicity results on elliptic problems with either Dirichlet or Robin boundary conditions and involving a general operator in divergence form. Moreover, some contributions for nonlinear problems involving a general operator not in divergence form are contained in [17, 18, 24]. Finally, our abstract methods can be also used studying fractional laplacian equations. See, for instance, the manuscript [25] and references therein for related topics.

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